Holonomy and nonholonomy in the dynamics of articulated motion

Pierre-Brice Wieber

INRIA Rhône-Alpes, 38334 St-Ismier Cedex, France
pierre-brice.wieber@inrialpes.fr

Summary. Walking, running or jumping are special cases of articulated motions which rely heavily on contact forces for their accomplishment. This central role of the contact forces is widely recognized now, but it is rarely connected to the structure of the dynamics of articulated motion. Indeed, this dynamics is generally considered as a complex nonlinear black-box without any specific structure, or its structure is only partly uncovered. We propose here to precise this structure and show in details how it shapes the movements that an articulated system might realize. Some propositions are made then to improve the design of control laws for walking, running, jumping or free-floating motions.

Key words: Gauss's principle, structure of the dynamics, articulated locomotion, nonholonomy.

1 Introduction

Improving the technology of sensors, actuators, computing power, mechanical design, might still be necessary in order to achieve faster and more reliable motions than what can be observed today in humanoid robotics, but better control law designs will probably be one of the key points. The design of feedback laws heavily relies on the understanding that we have of the underlying dynamics, and there may still be room for improvement here.

Walking, running or jumping are special cases of articulated motions that strongly depend on contact forces for their accomplishment. This central role of the contact forces is widely recognized now, but it is rarely connected to the structure of the dynamics of articulated motion. Indeed, this dynamics is generally considered as a complex nonlinear black-box without any specific structure, or its structure is only partly uncovered. We propose here to precise this structure and show in details how it shapes the movements that an articulated system might realize.

The existence of a structure in the dynamics of articulated motion has often been recognized [4, 7, 21, 25, 29], but it has never been analyzed as thoroughly as in [18]. Even in a study as precise as what can be found in [21, 20], it is not completely clear that behind the "d'Alembertian wrench" of the system studied there lies in fact its Newton and Euler equations, as will be stated here. Most of all, the holonomy and nonholonomy of the kinetic momenta and their implications for the locomotion of articulated systems has been rarely if not never discussed outside the works of the authors of [18], with the only exception of space robotics [16, 19].

The analysis that we are going to propose here is therefore deeply inspired by what can be found in [18]. Now, this brilliant work has been made in the framework of Lie algebras, a very powerful framework for high-level analysis of dynamical systems, but which may hide somehow the details appearing in the "real" equations to the reader who doesn't speak this language fluently. The main point of the present article is therefore to rederive these results without the use of Lie algebras. Doing so calls for an unusual way to derive the dynamics of articulated systems, through the use of Gauss's principle.

We're going therefore to present this principle and how it can be used to derive the dynamics of articulated systems in section 2. This original way of deriving this dynamics will be helpful then in section 3 in order to precise its inner structure. What this structure implies for the movements that articulated systems can realize will be studied then in section 4, where nonholonomy makes its first appearance. Nonholonomy will be the main topic then of section 5, where some implications of this phenomenon for the locomotion of articulated systems are put to light, and where propositions are made to make use of it in order to improve the design of control laws for walking, running, jumping or free-floating motions

2 Gauss's principle and the dynamics of articulated motion

2.1 Gauss's principle

Gauss's principle, equivalent to d'Alembert's one, can be seen as an extension of the principle of virtual work to the dynamical case. It states that the acceleration of a set of solids subject to some constraints deviates the least possible from the acceleration that it would have had without the constraints [23, 27]. This deviation is measured with the following kinetic metric:

$$\mathcal{D} = \sum_k \frac{1}{2}\,(\ddot{x}_k - \underline{\ddot{x}}_k)^T m_k\,(\ddot{x}_k - \underline{\ddot{x}}_k) + \frac{1}{2}\,(\dot{\omega}_k - \underline{\dot{\omega}}_k)^T\,\mathbb{I}_k\,(\dot{\omega}_k - \underline{\dot{\omega}}_k), \qquad (1)$$

with $\ddot{x}_k$ and $\dot{\omega}_k$ the translation and rotation accelerations of the k^{th} solid, m_k its mass, $\mathbb{I}_k$ its inertia matrix expressed at its center of mass, and $\underline{\ddot{x}}_k$ and $\underline{\dot{\omega}}_k$ the translation and rotation accelerations that it would have had without the constraints, that is the solutions of the classical Newton and Euler equations,

$$\begin{aligned} m_k\, \underline{\ddot{x}}_k &= f_k, \\ \mathbb{I}_k\, \underline{\dot{\omega}}_k - (\mathbb{I}_k\, \omega_k) \times \omega_k &= \tau_k, \end{aligned} \tag{2}$$

where f_k and τ_k are the forces and torques acting on this solid. Note that the Euler equation is expressed in a frame attached to the solid, as well as the velocity ω_k, reason why there is a gyroscopic term $(\mathbb{I}_k\, \omega_k) \times \omega_k$.

2.2 The dynamics of articulated motion

Considering now a set of solids constrained to move together by a set of articulations, their dynamics can be computed with the help of Gauss's principle. The constraints induced by the articulations can be expressed implicitly by describing the positions of the different solids of the system in a compact way through a configuration vector $q \in \mathbb{R}^n$. Their velocities and accelerations can be related then to the vectors $\dot{q}$ and $\ddot{q}$ with the help of translation and rotation jacobians:

$$\begin{aligned} \dot{x}_k &= J_{tk}(q)\, \dot{q}, \\ \omega_k &= J_{Rk}(q)\, \dot{q} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \ddot{x}_k &= J_{tk}(q)\, \ddot{q} + \dot{J}_{tk}(q, \dot{q})\, \dot{q}, \\ \dot{\omega}_k &= J_{Rk}(q)\, \ddot{q} + \dot{J}_{Rk}(q, \dot{q})\, \dot{q}. \end{aligned} \tag{4}$$

Introducing these relations in the definition (1) of the deviation $\mathcal{D}$ and solving the Newton and Euler equations (2) for $\underline{\ddot{x}}_k$ and $\underline{\dot{\omega}}_k$, the optimality condition for the minimization of this deviation turns into (we skip these calculations which are completely straightforward)

$$\frac{\partial \mathcal{D}}{\partial \ddot{q}} = M(q)\, \ddot{q} + N(q, \dot{q})\, \dot{q} - \mathcal{F} = 0, \tag{5}$$

with

$$M(q) = \sum_k J_{tk}^T\, m_k\, J_{tk} + J_{Rk}^T\, \mathbb{I}_k\, J_{Rk}, \tag{6}$$

$$N(q, \dot{q}) = \sum_k J_{tk}^T\, m_k\, \dot{J}_{tk} + J_{Rk}^T\, \mathbb{I}_k\, \dot{J}_{Rk} - J_{Rk}^T\, (\mathbb{I}_k\, J_{Rk}\, \dot{q}) \times J_{Rk}, \tag{7}$$

$$\mathcal{F} = \sum_k J_{tk}^T\, f_k + J_{Rk}^T\, \tau_k. \tag{8}$$

We end up therefore with a classical Lagrangian description of the dynamics of a system of articulated bodies with an inertia matrix $M(q)$, nonlinear dynamical effects $N(q, \dot{q})\, \dot{q}$ and generalized forces $\mathcal{F}$ acting on the system. Note that we're taking some liberties in (7) and in the following with the notation of the cross-product by considering that given a vector $v \in \mathbb{R}^3$, the notation $(v)\times$ means in fact multiplying by the classical anti-symmetric matrix

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

The point here is that even though the definition (6) of the inertia matrix is identical to what can be found in usual robotics textbooks [15, 26], it is not the case for the definition (7) of the nonlinear effects. Indeed, these nonlinear effects are generally presented through Christoffel symbols that completely hide the structure that can be seen here, and this structure is going to be very useful in analyzing the dynamics of articulated motion. For example, one can observe immediately with the definitions here that the matrix $\dot{M} - 2N$ is anti-symmetric.

3 Inner structure of the dynamics of articulated motion

3.1 The structure of the configuration vector

In the case of locomoting or free-floating articulated systems, the configuration vector $q \in \mathbb{R}^n$ introduced in the previous section stitches in fact together three very distinct informations,

$$q = \begin{bmatrix} \hat{q} \\ x_0 \\ \theta_0 \end{bmatrix}, \tag{9}$$

where $\hat{q}$ describes the positions of the articulations of the system, and x_0 and θ_0 the position and the orientation of a reference frame attached to one solid of the system.

This structure of the configuration vector can be found then in all the kinematic and dynamic equations of the system, to begin with the translation and rotation jacobians that were introduced in (3). Indeed, if we consider the translation and rotation velocities $\dot{\hat{x}}_k$ and $\hat{\omega}_k$ of the k^{th} solid with respect to the reference frame that we have just introduced, they must be composed with the velocities $\dot{x}_0$ and ω_0 of this reference frame itself in order to obtain the total translation and rotation velocities $\dot{x}_k$ and ω_k of the solid. This is done through the following classical composition rules (remember that the rotation velocities ω are expressed in local frames):

$$\begin{aligned} \dot{x}_k &= R_0(\theta_0)\,\dot{\hat{x}}_k + \dot{x}_0 + (R_0(\theta_0)\,\omega_0) \times (x_k - x_0), \\ R_k(q)\,\omega_k &= R_k(q)\,\hat{\omega}_k + R_0(\theta_0)\,\omega_0 \end{aligned}$$

with $R_0(\theta_0)$ and $R_k(q)$ the orientation matrices of the k^{th} solid and the reference frame with respect to the inertial frame. Now, if we use the fact that the velocities $\dot{\hat{x}}_k$ and $\hat{\omega}_k$ are solely related to the vector $\dot{\hat{q}}$,

$$\begin{aligned} \dot{\hat{x}}_k &= \hat{J}_{tk}(\hat{q})\,\dot{\hat{q}}, \\ \hat{\omega}_k &= \hat{J}_{Rk}(\hat{q})\,\dot{\hat{q}}, \end{aligned}$$

and the fact that the rotation velocity ω_0 of the reference frame can be related to the angular velocity $\dot\theta_0$,

$$\omega_0 = J_{R0}(\theta_0)\,\dot\theta_0,$$

these composition rules turn into (with shortened notations)

$$\dot x_k = R_0\,\hat J_{tk}\,\dot{\hat q} + \dot x_0 - (x_k - x_0) \times R_0\,J_{R0}\,\dot\theta_0,$$

$$\omega_k = \hat J_{Rk}\,\dot{\hat q} + R_k^T R_0\,J_{R0}\,\dot\theta_0,$$

where we can observe that the translation and rotation jacobians that were introduced in (3) exhibit a structure corresponding exactly to the structure (9) of the configuration vector:

$$\begin{aligned} J_{tk} &= \begin{bmatrix} R_0\,\hat J_{tk} & \mathbb{1}_{3\times3} & -(x_k - x_0) \times R_0\,J_{R0} \end{bmatrix} \\ J_{Rk} &= \begin{bmatrix} \hat J_{Rk} & \mathbb{0}_{3\times3} & R_k^T\,R_0\,J_{R0} \end{bmatrix} \end{aligned} \tag{10}$$

with $\mathbb{0}_{3\times3}$ and $\mathbb{1}_{3\times3}$ a zero and an identity matrix.

3.2 Back to Newton and Euler equations

Replacing this structure of the jacobians J_{tk}^T and J_{Rk}^T in (6)-(8), we obtain a structure of the inertia and non-linear effects matrices and of the generalized forces that corresponds once again exactly to the structure (9) of the configuration vector:

$$M(q) = \sum_k \begin{bmatrix} \hat J_{tk}^T\,R_0^T\,m_k\,J_{tk} + \hat J_{Rk}^T\,\mathbb{I}_k\,J_{Rk} \\ m_k\,J_{tk} \\ J_{R0}^T\,R_0^T\big((x_k - x_0) \times m_k\,J_{tk} + R_k\,\mathbb{I}_k\,J_{Rk}\big) \end{bmatrix}, \tag{11}$$

$$N(q,\dot q) = \sum_k \begin{bmatrix} \hat J_{tk}^T\,R_0^T\,m_k\,\dot J_{tk} + \hat J_{Rk}^T\,\mathbb{I}_k\,\dot J_{Rk} - \hat J_{Rk}^T\,(\mathbb{I}_k J_{Rk}\,\dot q) \times J_{Rk} \\ m_k\,\dot J_{tk} \\ J_{R0}^T\,R_0^T\big((x_k - x_0) \times m_k\,\dot J_{tk} + R_k\,\mathbb{I}_k\,\dot J_{Rk} - R_k\,(\mathbb{I}_k J_{Rk}\,\dot q) \times J_{Rk}\big) \end{bmatrix}, \tag{12}$$

$$\mathcal{F} = \sum_k \begin{bmatrix} \hat J_{tk}^T\,R_0^T\,f_k + \hat J_{Rk}^T\,\tau_k \\ f_k \\ J_{R0}^T\,R_0^T\big((x_k - x_0) \times f_k + R_k\,\tau_k\big) \end{bmatrix}. \tag{13}$$

The dynamics (5) can be split therefore in three lines, each one with a very specific structure. Particularly interesting are the two last ones: with the help of relations (3) and (4), the line in the middle gives

$$\sum_k m_k \ddot{x}_k = \sum_k f_k$$

and the last line gives

$$J_{R0}^T R_0^T \sum_k (x_k - x_0) \times m_k \ddot{x}_k + R_k \mathbb{I}_k \dot{\omega}_k - R_k (\mathbb{I}_k \omega_k) \times \omega_k = J_{R0}^T R_0^T \sum_k (x_k - x_0) \times f_k + R_k \tau_k$$

what, putting aside the multiplication by $J_{R0}^T R_0^T$, corresponds to an equality between the dynamical momentum of rotation of the whole system and the sum of all the torques applied to it, both expressed with respect to x_0 in an absolute reference frame. What appears here are therefore simply a Newton and an Euler equation for the whole system.

3.3 Forces acting on the system

We're going to consider three different types of generalized forces, those which are most generally found acting on systems of articulated bodies: the gravity $\mathcal{F}_g$, the control forces $\mathcal{F}_u$ and the contact forces $\mathcal{F}_c$. The gravity forces and torques acting on each solid are simply $f_k = m_k g$ and $\tau_k = 0$, where g is simply the vector of the gravity field. Replacing this in (13) gives

$$\mathcal{F}_g = \sum_k \begin{bmatrix} \hat{J}_{tk}^T R_0^T m_k g \\ m_k g \\ J_{R0}^T R_0^T (x_k - x_0) \times m_k g \end{bmatrix}. \tag{14}$$

Now, we can observe from the structure (10) of the translation and rotation jacobians that we obviously have

$$J_{tk} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = g \quad \text{and} \quad J_{Rk} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = 0$$

so that a short inspection of (11) leads to the fact that

$$\mathcal{F}_g = M(q) \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}.$$

This implies that the dynamics (5) of the system under the action of gravity can be written as:

$$M(q) \left(\ddot{q} - \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \right) + N(q, \dot{q}) \dot{q} = 0.$$

Here lies the obvious observation that even on an articulated system, the action of gravity is nothing but a linear acceleration in the direction of the gravity field.

Concerning control forces, we'll consider that the only ones acting on the system are internal forces acting between the different solids of the system, coming from the action of muscles or actuators on the articulations of the system. In this case, the application of Newton's law of action and reaction leads us immediately to the conclusion that their sum (13) is of the form

$$\mathcal{F}_u = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

and we're not going to precise any deeper the structure of the vector u for the analysis undertaken here.

Concerning the contact forces between the system and its environment, very different models exist [3] and we will focus only on the fact that whatever the model, there are always similar constraints on their direction and amplitude due to unilaterality and limited friction. We will gather all these limitations in a vector inequality relating these forces to the position of the system:

$$\mathcal{A}(q, \mathcal{F}_c) \leq 0 \tag{16}$$

4 Motions that an articulated system can realize

4.1 With only control forces

If we introduce the center of mass x_G of the system,

$$x_G = \frac{1}{m} \sum_k m_k x_k \quad \text{with} \quad m = \sum_k m_k,$$

the Newton equation of the whole system appears to be simply

$$m\, \ddot{x}_G = \sum_k f_k. \tag{17}$$

If we get rid of the multiplication by $J_{R0}^T\, R_0^T$ in the Euler equation and if we add to it

$$(x_0 - x_G) \times \sum_k m_k \ddot{x}_k = (x_0 - x_G) \times \sum_k f_k$$

in order to express the momentum of rotation and the sum of torques with respect to the center of mass instead of x_0, we obtain

$$\sum_k (x_k - x_G) \times m_k \ddot{x}_k + R_k\, \mathbb{I}_k\, \dot{\omega}_k - R_k\, (\mathbb{I}_k\, \omega_k) \times \omega_k = \sum_k (x_k - x_G) \times f_k + R_k\, \tau_k, \tag{18}$$

what is equal to

$$\frac{d}{dt}\left[\sum_k (x_k - x_G) \times m_k \dot{x}_k + R_k \, \mathbb{I}_k \, \omega_k\right] = \sum_k (x_k - x_G) \times f_k + R_k \, \tau_k,$$

where the left hand side appears to be simply the derivative of the kinetic momentum of rotation of the system. This way, we can see that if the system is under the action of only the control forces (15), we have the obvious conservation of the kinetic momenta:

$$\begin{aligned} m \, \dot{x}_G &= \text{Constant}, \\ \sum_k (x_k - x_G) \times m_k \dot{x}_k + R_k \, \mathbb{I}_k \, \omega_k &= \text{Constant}. \end{aligned}$$

If the system starts with a zero velocity, these constants are zero and the first equation implies that whatever the control forces employed, the center of mass of the system will remain unmoved. The implications of the second equation are more subtle since it is a nonholonomic constraint, a relation between the velocities of the bodies of the system that doesn't imply a relation between their positions and orientations: it constrains the movements that the system can realize, but not the positions that it can reach.

We're going to spend more time in section 5 on the many implications of this nonholonomy, but we can already stress that notwithstanding this conservation of the kinetic momentum of rotation, the position of the articulations $\hat{q}$ and the orientation of the system θ_0 can be controlled together to any desired value, with the only action of muscles or actuators on the articulations of the system: controlling the articulations of the system is enough to control also its orientation.

4.2 With gravity forces

If the system is under the action of the gravity forces (14) in addition to the control forces considered earlier, we can observe that the only modification to the movements of the system is that its center of mass will be linearly accelerated along the gravity vector instead of staying idle:

$$\ddot{x}_G = g.$$

The conservation of the kinetic momentum of rotation is unchanged, and so is the conclusion about its nonholonomy, and so is therefore the fact that controlling the articulations of the system is enough to control also its orientation. This can be observed in the most usual example of articulated system under the action of gravity and muscles, a cat falling which always manages to fall back on its feet. Figure 1 shows the similar case of a dog, and a close inspection of this stop-motion allows to understand how the rotation of the

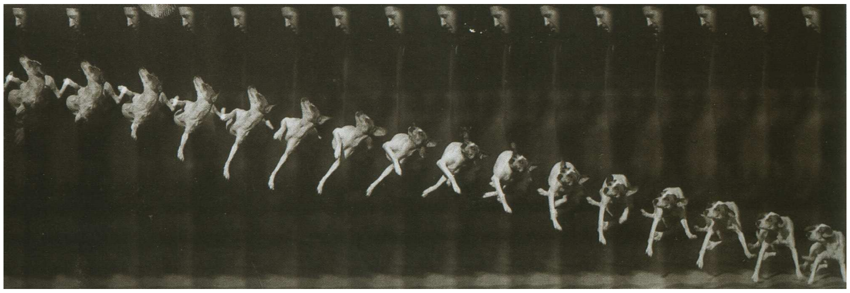

Fig. 1. Falling back on the feet thanks to nonholonomy [6].

body is undertaken even though the kinetic momentum of rotation is kept unchanged to zero: limbs are moved back and forth with different lengths, and therefore with different inertial properties. We're going to focus again later on this very simple principle.

4.3 With contact forces

Of the three types of forces considered in section 3.3, the contact forces appear therefore to be the only one able to generate movements of the center of mass of the system in any way other than falling down. The locomotion of articulated systems completely rely therefore on the availability of contact forces. Now, we have seen that these contact forces are limited because of unilaterality and limited friction, as has been resumed in the general inequality (16). Every movement undertaken by an articulated system has therefore to comply with these limitations.

Research in biped locomotion has been extensively focusing on this question, and in different ways, either focusing explicitly on the dynamic momenta that appear in the Newton and Euler equations (17)-(18) as in the Resolved Momentum Control approach [10, 11], in the Zero Moment Point analysis [8, 28] and in other similar works [9, 13, 22], or treating more globally the force allocation problem directly in the Lagrangian equation (5) as in [7, 14, 29]. Since we have seen that this Lagrangian equation embeds explicitly the Newton and Euler equations of the system, we can conclude that these two ways of approaching the problem are exactly equivalent.

The contact forces are also the only ones which can have an effect on the momentum of rotation of the system and therefore potentially remove, within the bounds of the inequalities (16), all the dynamical constraints that existed on the movements of articulated systems. This point is less crucial though than the problem of moving the center of mass since we have seen that the nonholonomy of this momentum of rotation allows for a control of the orientation of the system through the control of its articulations. This is going to be the main topic of the next section.

Fig. 2. 22 degrees of rotation of the whole body in the sagittal plane induced by 24 steps of the walking pattern recorded in [31] and replayed on a simple free-floating biomechanical model, what corresponds to a complete turn in 393 steps.

5 Some implications on the locomotion of articulated systems

5.1 Nonholonomy of the momentum of rotation when walking

We have seen in section 4.2 that the nonholonomy of the momentum of rotation can be observed in the case of the very specific movements that cats and dogs realize in order to fall back on their feet. It is in fact a very general phenomenon that can be observed even when walking. When replaying for example the walking pattern recorded in [31] on a simple biomechanical model with no external forces and therefore with a kinetic momentum of rotation constantly equal to zero, we can indeed observe a rotation of the whole body happening in the sagittal plane (Fig. 2). Note that a general property of such nonholonomic constraints is that the outcome of the movement doesn't depend on its speed, but solely on its shape, so we can measure the nonholonomy here as an amount of rotation of the whole body for each step of walking, depending solely on the shape of this step, regardless of its actual speed. In this case, that gives $1/393^{rd}$ of a complete turn of the body per step accomplished.

A more accurate measurement of this phenomenon can be obtained on a robotic system such as the HRP-2 robot, for which the inertial properties and the movements actually realized can be known with very good precision. We

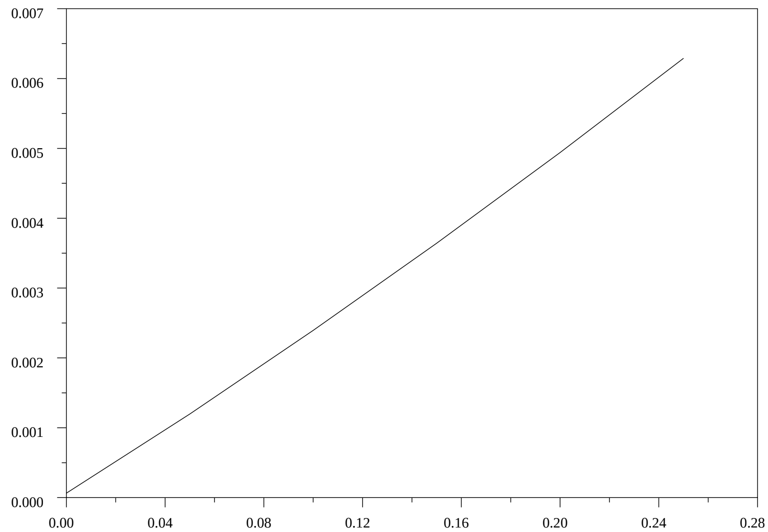

Fig. 3. Fraction of a complete turn in the sagittal plane of the whole body of the HRP-2 robot induced by a step of walking as a function of the height of this step (in meters), with up to $1/159^{th}$ of a complete turn for a 25 cm high step.

can accurately measure then the amount of rotation of the whole body as a function of the shape of the step, for example its height in Fig. 3, reaching here $1/159^{th}$ of a complete turn of the body for a 25 cm high step.

Note that this phenomenon is intrinsic to the movements of the legs when walking, back and forth with different lengths in order to avoid undesired contacts with the ground, and therefore back and forth with different inertial properties, just as what has been observed in the case of the dog falling in Fig. 1. Keeping the body upright when walking necessarily calls therefore for a second phenomenon in order to counterbalance this rotation. One can imagine the arms making the exact inverse of the movements of the legs, with different lengths when moving back and forth, but this is not what can be observed in natural walking patterns, without mentionning the cases when the arms don't even move back and forth, when holding a heavy object or when keeping arms crossed. The counterbalancing phenomenon that can be observed is in fact a non-zero mean kinetic momentum of rotation in the direction opposite to the rotation induced by the legs' movements (Fig. 4). With these observations in mind, the proposition made in [22] of controlling the kinetic momentum of rotation of a walking system to keep it to zero appears to be problematic: a zero kinetic momentum of rotation appears to be incompatible with walking,

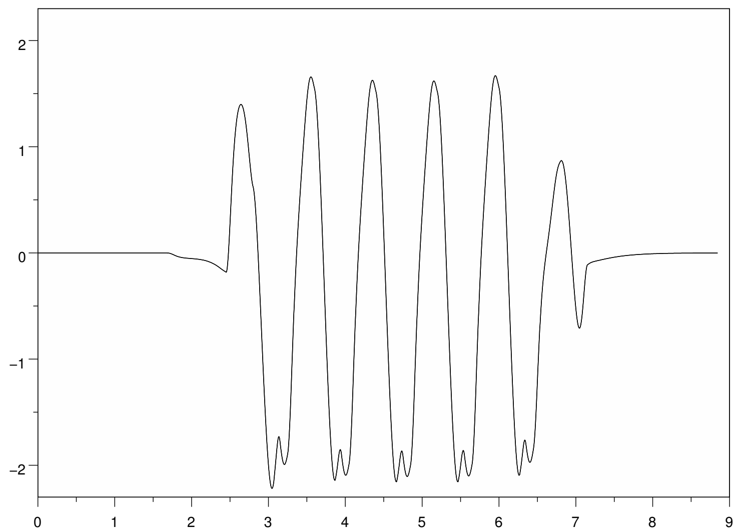

Fig. 4. Kinematic momentum of rotation of the HRP-2 robot in the sagittal plane (in kg.m^2.s^{-1}) as a function of time (in seconds) when executing 25 cm high steps on the ground.

and designing a control law to do so will obviously lead to an error either in the tracking of the walking movements or in the tracking of the zero kinetic momentum, both potential sources of instability.

It could be tempting then to control this kinetic momentum of rotation to a non-zero value, following for example the framework presented in [11], but the choice of the desired value may not be easy to decide since it should be made according to the specific shape of the walking pattern being considered. On top of that, the kinetic momentum of rotation can be observed to vary strongly when walking (Fig. 4), so the choice of a constant value might be an unnecessary limitation. But most of all, as we have already observed in Fig. 2, the value of the kinetic momentum of rotation is only scarcely related to the actual rotations that the articulated system is going to realize: deciding a value for this momentum decides in fact almost nothing about the movement to come. It may not be sure therefore whether focusing specifically on the control of the kinetic momentum of rotation as proposed in [1, 9, 11, 13, 22] is the best option, after all. At least, it is incomplete in controlling the motion of an articulated system.

5.2 Some hints for improving the control of walking, running, jumping and free floating motions of articulated systems

The amounts of rotation measured in the previous section, $1/393^{rd}$ to $1/159^{th}$ of a complete turn, might look small if not negligible, the main reason that led to the erroneous conclusion in [22] that they might be strictly zero. But we should not forget the example of the cat always falling back on its feet: nonholonomy can be a very precious tool in the control of the locomotion of articulated systems, not to be underestimated.

Of course, solely varying the height of the steps as in the previous section is not a serious solution to completely stabilize a walking movement, but this is an indication that varying the shape of the walking pattern can help improve this stability, especially if motions of the whole body are involved, for greater efficiency (the arms can be a precious source of inertial effects). At least, this can be a valuable addition to the methods already known for stabilizing walking motions such as varying the step lengths or the speed [30].

Useful in the case of walking, making use of this nonholonomy can become an absolute necessity in the case of running, jumping and free floating motions, being the only way to control the orientation of the system when contact forces are not available anymore. Now, it is well known as a side effect of a famous theorem due to R.W. Brockett [2] that the complete control of a system with nonholonomic constraints can't be realized with continuous time-invariant feedback control laws: discontinuous or time-varying control laws are a necessity in this case [5]. This explains why the use of this nonholonomy is out of reach of the control laws proposed in [12, 17, 25] which are all continuous and time-invariant.

Using this nonholonomy is a well established control method in space robotics [16, 19, 24], but the only control law making such a use of this nonholonomy that seems to have been proposed so far for humanoid locomotion is for running, in [4]. There, a time-varying control law is proposed in the flight phase by simply letting an additional degree of freedom in the design of the trajectories be used to control the orientation of the system. We can observe there that calling for discontinuous or time-varying control laws doesn't necessarily imply complex solutions: local modifications of the shape of the limb trajectories can be just enough, what can be made even easier from a computational point of view with the help of a library of precomputed motions.

6 Conclusion

The core of this short note has been a precise description of the inner structure of the dynamics of articulated systems of bodies, establishing in particular how the Newton and the Euler equations of the whole system are very simply and directly embedded inside its Lagrangian dynamics, implying an immediate equivalence between the approaches that focus on the former equations and the approaches that focus on the latter one. This whole analysis has been made possible in the first place thanks to an original derivation of this Lagrangian dynamics through the use of Gauss's principle.

Conclusions have been derived then concerning the holonomy of the Newton equation and the nonholonomy of the Euler equation, implying the necessity of contact forces for articulated systems to realize translations, but not rotations for which joint forces are enough. A specific analysis of this latter phenomenon has been undertaken then in the case of walking motions, and propositions have been finally made to make use of it for improving the design of control laws for stabilizing walking, running, jumping, and more generally every kind of articulated movements on the ground and in the air.

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